TAUTOCHRONE BALLS: A MATHCAD ANIMATION

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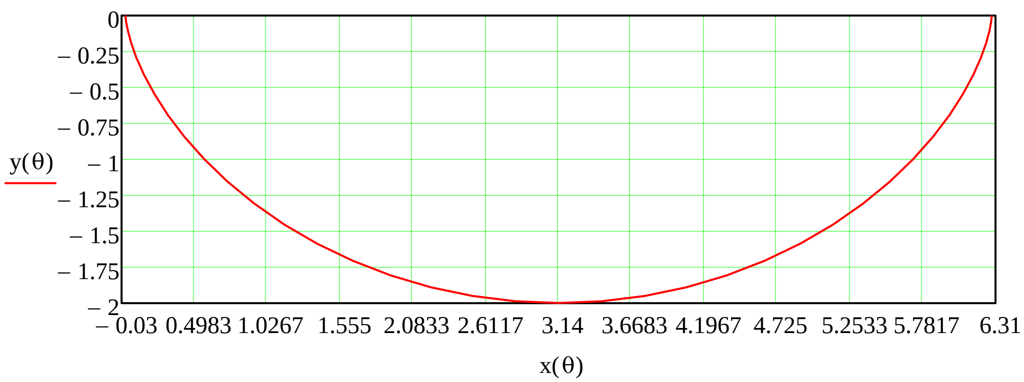
Of all the curves that can be drawn between points (0,0) and (π,-2) in the xy plane, a cycloid is the path down which a Newtonian particle will fall fastest. The Swiss mathematician Johann Bernoulli discovered this "brachistochrone" property of the cycloid in 1696 (brachistochrone is Greek for "fastest time"). Bernoulli's discovery, and the challenge of its proof, led to a new mathematical discipline, the variational calculus, or "calculus of variations."

But Christiaan Huygens, the Dutch mathematician, physicist, astronomer and clockmaker, had already discovered in 1658 that the cycloid is a "tautochrone" (Greek for "same time"), in that a particle will fall to the bottom of the cycloid in the same amount of time, no matter where it is placed. It is the tautochrone property of the cycloid that is the basis for this worksheet and its "Tautochrone Balls" animation.

Here are the parametric equations of the cycloid, and a plot of same for a = 1:

$\theta := 0, \frac{\pi}{20} .. 2\cdot\pi$ $\qquad$ $a := 1m$

$x(t) := a\cdot(t - \sin(t))$ $\qquad$ $y(t) := -a\cdot(1 - \cos(t))$



The cycloid gets its name from the fact that it is the curve swept out by any point fixed on the rim of a rolling wheel. In the graph above, the center of the rolling wheel (i.e., a circle of radius a = 1) moves along the line y = -1. As also can be seen from the graph, the generating point on the rim of the wheel is at x = 0, y = 0 when the wheel starts to roll. Many books have been published that show how to use the calculus of variations to prove that the curve of fastest descent of a particle is a cycloid.

But those that I have in my own library stop the analysis when the form of the curve is known. What I needed to know was, how does θ depend upon time? More specifically, does the generating circle roll at a constant rate?

The answer to the second question is yes. The answer to the first question is obtained as follows:

$$\left(\frac{d}{dt}x\right)^2 = \left[a\cdot(1-\cos(\theta))\cdot\left(\frac{d}{dt}\theta\right)\right]^2 = y^2\cdot\left(\frac{d}{dt}\theta\right)^2$$

and

$$\left(\frac{d}{dt}y\right)^2 = \left[a\cdot\sin(\theta)\cdot\left(\frac{d}{dt}\theta\right)\right]^2 = a^2\cdot\sin(\theta)^2\cdot\left(\frac{d}{dt}\theta\right)^2.$$

Noting that

$$y^2 = a^2 - 2\cdot a^2\cdot\cos(\theta) + a^2\cdot\cos(\theta)^2,$$

we have

$$y^2 + a^2\cdot\sin(\theta)^2 = 2\cdot a^2 - 2\cdot a^2\cdot\cos(\theta) = 2\cdot a\, y.$$

Thus

$$\left(\frac{d}{dt}x\right)^2 + \left(\frac{d}{dt}y\right)^2 = 2\cdot a\cdot y\cdot\left(\frac{d}{dt}\theta\right)^2.$$

But since

$$\left(\frac{d}{dt}x\right)^2 + \left(\frac{d}{dt}y\right)^2 = \left(\frac{d}{dt}s\right)^2 = 2\cdot g\cdot y \quad \text{(see [1, pp. 482-484])},$$

where s is arc length along the curve, and g is the gravitational constant assumed to be 9.807 m/s^2, we have

$$2\cdot a\cdot y\cdot\left(\frac{d}{dt}\theta\right)^2 = 2\cdot g\cdot y$$

So $\frac{d}{dt}\theta = \sqrt{\frac{g}{a}}$. The cycloid-generating circle does indeed roll at a constant rate, and we have $\theta = \sqrt{\frac{g}{a}}\cdot t$ for $0 \le t \le 1$.

We also see that T, the period of the motion, for which $\theta = 2\cdot\pi$, is:

$T := \sqrt{\frac{a}{g}} \cdot 2 \cdot \pi$ $\quad T = 2.006\,s \quad g := 9.807 \quad a := 1$

That is, for a generating circle of radius of a = 1 m, it takes about 2 seconds for the circumferential point to travel back to the height at which it started. This corresponds to a tautochrone ball falling to the lowest point on the curve and then returning back to the height at which it started.

Also, the center of the generating circle travels 2π meters during this period of time. This can be seen from the graph.

We now set up the data needed to plot and animate four tautochrone balls.

First we set up the cycloid curve itself, in green.

$t_1 := 0, \frac{1}{100} .. 1.0 \qquad t_2 := t_1$

$t_1 := \text{if}\left(\text{FRAME} \le 100, \frac{\text{FRAME}}{100}, -\frac{\text{FRAME} - 100}{100}\right)$ FRAME variable is for animation.

$x_1(t_1) := a \cdot \left(2 \cdot \pi \cdot t_1 - \sin(2 \cdot \pi \cdot t_1)\right)$

$y_1(t_1) := -a \cdot \left(1 - \cos(2 \cdot \pi \cdot t_1)\right)$

We now set three values, p1, p2, and p3, all between 0 and 0.5, so as to plot three balls, none which of starts at the top of the tautochrone curve .

<u>Ball 2</u>

(red)

$p1 := \frac{10}{50} \qquad y_1(p1) = -0.691$

$\theta 1 := \text{acos}\left(1 + \frac{y_1(p1)}{a}\right) \qquad t1 := \sqrt{\frac{a}{g}} \cdot \theta 1$

$\theta 1 = 1.257 \qquad t1 = 0.401$

$x_2(t_1) := \text{if}\left[\text{FRAME} \le 100, x_1\left[p1 + (1 - t1) \cdot t_1\right], x_1\left[1 - p1 + (1 - t1) \cdot t_1\right]\right]$

$y_2(t_1) := \text{if}\left[\text{FRAME} \le 100, y_1\left[p1 + (1 - t1) \cdot t_1\right], y_1\left[1 - p1 + (1 - t1) \cdot t_1\right]\right]$

Ball 3

(blue)

$$p2 := \frac{15}{50} \qquad y_1(p2) = -1.309$$

$$\theta 2 := \mathrm{acos}\left(1 + \frac{y_1(p2)}{a}\right) \qquad t2 := \sqrt{\frac{a}{g}}\cdot\theta 2$$

$$\theta 2 = 1.885 \qquad t2 = 0.602$$

$$x_3(t_1) := \mathrm{if}\left[FRAME \le 100, x_1\left[p2 + (1 - t2)\cdot t_1\right], x_1\left[1 - p2 + (1 - t2)\cdot t_1\right]\right]$$

$$y_3(t_1) := \mathrm{if}\left[FRAME \le 100, y_1\left[p2 + (1 - t2)\cdot t_1\right], y_1\left[1 - p2 + (1 - t2)\cdot t_1\right]\right]$$

Ball 4

(brown)

$$p3 := \frac{20}{50} \qquad y_1(p3) = -1.809$$

$$\theta 3 := \mathrm{acos}\left(1 + \frac{y_1(p3)}{a}\right) \qquad t3 := \sqrt{\frac{a}{g}}\cdot\theta 3$$

$$\theta 3 = 2.513 \qquad t3 = 0.803$$

$$x_4(t_1) := \mathrm{if}\left[FRAME \le 100, x_1\left[p3 + (1 - t3)\cdot t_1\right], x_1\left[1 - p3 + (1 - t3)\cdot t_1\right]\right]$$

$$y_4(t_1) := \mathrm{if}\left[FRAME \le 100, y_1\left[p3 + (1 - t3)\cdot t_1\right], y_1\left[1 - p3 + (1 - t3)\cdot t_1\right]\right]$$

Ball 1

(black)

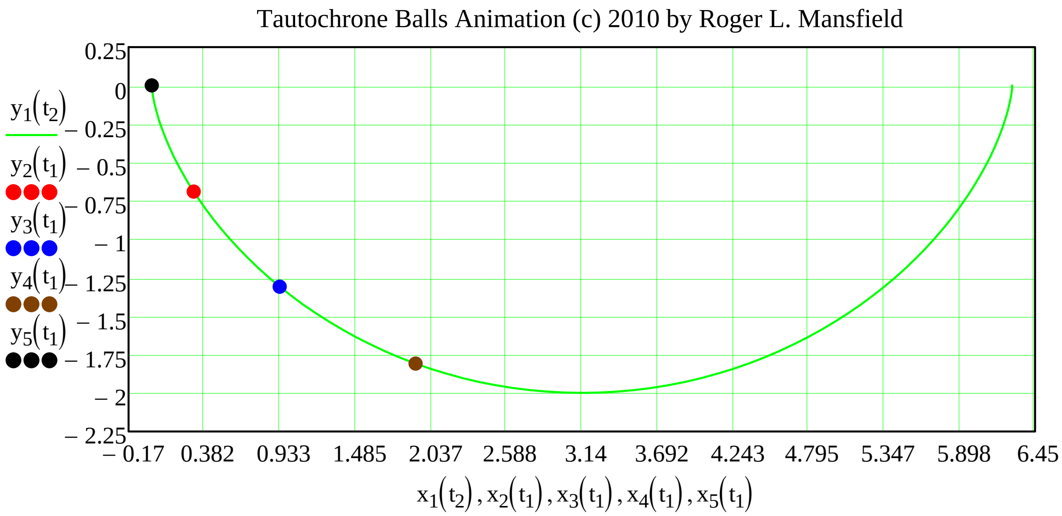
$$x_5(t_1) := \mathrm{if}\left(FRAME \le 100, x_1(t_1), x_1(1 + t_1)\right)$$

$$y_5(t_1) := \mathrm{if}\left(FRAME \le 100, y_1(t_1), y_1(1 + t_1)\right)$$

Note that ball 1 (black) starts at the top of the cycloid curve, at x = 0 and y = 0, while ball 2 (red), ball 3 (blue) and ball 4 (brown) all start farther down.

Mathcad Animation

To animate the plot below, select Tools>Animation>Record, set FRAME to go from 0 to 200, drag/select the plot, and click on Animate.



The curve is a cycloid, so all four balls fall to the bottom of the curve, at x = π and y = -2, in the same amount of time!

REFERENCES

[1] Boas, Mary L., *Mathematical Methods in the Physical Sciences, 3rd Edition* (John Wiley, 2006). See Chapter 9, Calculus of Variations, pp. 482-484.

[2] Boyer, Carl B., *A History of Mathematics, 2nd Edition* (John Wiley, 1991). Revised by Uta C. Merzbach. See pp. 374-378.

[3] Elsgolc, L. E., *Calculus of Variations* (Addison-Wesley, 1961). Reprinted by Dover, Inc., and still in print (178 pages).

[4] Thomas, George B. Jr. and Finney, Ross L., *Calculus and Analytical Geometry, 8th Edition* (Addison-Wesley, 1992). Proof that the brachistochrone (cycloid) is also a tautochrone is given on pp. 654-655.