BATCH LEAST SQUARES DIFFERENTIAL CORRECTION
OF A HELIOCENTRIC ORBIT

PART 2 - MANUAL CORRECTION WORKSHEET

Roger L. Mansfield, June 21, 2016
http://astroger.com

In this worksheet we differentially correct (DC) the orbit of a comet, minor planet, or interplanetary space probe using a test case specified in worksheet HD1, or in a worksheet derived from HD1. You should open worksheet HD1, or your own worksheet derived from HD1, and click on "Calculate Worksheet" from the Math menu now, if you have not already done so.

The process that we will use in this worksheet is documented in Refs. [1], [2], and [9] for the differential correction of Earth orbits. We will use only optical (astrometric) observations in this worksheet. The batch equation of differential correction (BEDC) is:

$$X_o' = X_o + (A^TWA)^{-1} A^TW [Y - F(X_o)].$$

Here X_o is the initial estimate of the state vector, i.e., position and velocity, at epoch t_o. X_o' is the "improved" estimate of X_o at t_o, obtained by adding to X_o the correction $(A^TWA)^{-1} A^TW [Y - F(X_o)]$.

If we let n be the number of observations, then Y is a 2n-by-1 column vector of measurements, since for our problem in heliocentric motion the measurements are topocentric right ascension (RA, or α) and topocentric declination (DEC, or δ). If we denote the total number of measurements by N, then N = 2n.

$F(X_o)$ is thus an N-by-1column vector of "computed" measurements. What this means is that the RA and DEC for each observation are computed via our UPM model of two-body motion, by propagating the current estimate, X_o, to the observation times t_i for i = 1, ... , n, and by then computing the topocentric RA and DEC at each observation time, given the specified location of the observer. We say "current estimate, X_o" because we will find it necessary to iterate on the BEDC, testing for convergence at each iteration by means of a criterion we will define below. If we have convergence on a given iteration, then we stop and convert the solution to conic elements. But if we do not have convergence, then we replace X_o by X_o' and solve the BEDC again, i.e., iterate. (We could also implement an iteration counter and stop the DC if some maximum allowable number of iterations is reached without convergence, but that is not needed here because we iterate the BEDC manually by clicking on "Calculate Worksheet".)

$[Y - F(X_o)]$ is the N-by-1column vector of residuals, in the sense "observed minus computed". The BEDC is a form of the least squares normal equations, N equations in six unknowns, which result when one answers the question, "what is a necessary condition that the weighted sum of squares of the residuals be a minimum?" The residuals are not actually $\Delta\alpha$ and $\Delta\delta$, but rather cos δ $\Delta\alpha$ and $\Delta\delta$; they are the projections of $\Delta\mathbf{L}$ on **A** and **D** in turn. (The cos δ factor can become quite important when the object passes near a celestial pole, where large changes in α accompany relatively small changes in arc length in the direction of motion.)

TIPS ON READING MATHCAD WORKSHEETS

1. A Mathcad worksheet typically consists of text regions and math regions.

2. Text and math regions can be anywhere on a page. Text regions are just optional comments. Mathcad uses math regions to do its calculations, and according to the following rule: math regions are calculated in the order of left to right, then top to bottom.

3. In a math region, colon-equals (:=) is like an assignment statement in C. That is, the expression on the right is calculated and placed in the variable on the left, whereas equals (=) by itself is used to display on the right of the equals sign the value of the variable on the left.

4. Mathcad has functions just like C does. Typically a function has its name and input variables inside of parentheses on the left of a colon-equals (:=) and a sequence of vertical line segments with the assignment statements of the function on the right. Inside of a function, the assignment statements use left arrows <-- instead of colon-equals := to make assignments. The last line of a function is its output argument, which may be a a scalar, a vector, or a matrix of variables.

5. My Mathcad worksheets generally are formatted so that the flow is left to right, top to bottom of an 8.5" x 11" page when the workshet is printed.

- But I like to use a second 8.5" page side-by-side for additional material, added later, that I might later delete, e.g., scratchpad or temporary calculations.

- Sometimes I use this extra right-margin space to do, in parallel, calculations needed later on, or to simply add material without disturbing the main flow.

- When my worksheet has the second, side-by-side page and I want to publish the worksheet, I print the worksheet as an Adobe .pdf file. I specify the "ledger" format, which, thankfully, prints out both pages side-by-side.

These tips themselves fit my second, side-by-side page convention: they are additionaly material that is not part of the main flow of the worksheet.

A, the "A-matrix", is the N-by-6 array of partial derivatives of the N measurements with respect to the six components of the state vector X_o. We will compute the A-matrix from the O-matrix and the G-matrix, i.e., A = OG. O is the N-by-6 matrix of partials of the measurements with respect to the state vector at observation times t_i, for i = 1, ... , n. G is Goodyear's 6-by-6 state transition matrix, i.e., the 6-by-6 matrix of partials of the state components at times t_i with respect to the state components at t_o. G is therefore a 6-by-6 Jacobian matrix defined at each observation time t_i, for i = 1, ... , n.
W is the weight matrix. Under the assumption that the measurements are Gaussian random variables, and are not correlated (Danby [3] has a good discussion of this), W is a diagonal matrix and each diagonal entry is $1/\sigma_i^2$, where σ_i^2 is the variance of measurment i. (We implement W here only for completeness; we will take W as the N-by-N identity matrix in this worksheet.)

Here now is an outline of the steps we will follow:

1. Retrieve the test case values from disk, as specified by worksheet HD1, or as specified by your own worksheet that was derived from HD1 by duplication and modification.

   Retrieval includes obtaining the initial or current estimate of state, X, and the RMS history matrix. Each time you click on "Calculate Worksheet", HDC performs another iteration of weighted, batch least squares differential correction. At each iteration the corrected values of X are written to disk along with the RMS for that iteration. The corrected values of X thus become the current state estimate for the next iteration, and the RMS history is accumulated so that you can keep track of how the DC is going.

2. Define the procedural functions needed in the DC: **C**, **FG**, **GMAT**, and **FXA**.

3. Obtain the computed measurements, **FX**, and the A-matrix, A, by invoking **FXA**.

4. Compute the residuals, ΔY, the A^TWA matrix ATWA, and the $A^TW\Delta Y$ matrix, ATWΔY.

5. Solve for and apply the corrections to state, ΔX. Compute the current RMS, display the RMS history, and test for convergence.

6. Write the corrected state vector to disk and convert to conic elements.

7. Repeat Steps 1-6, by clicking on "Calculate Worksheet", until convergence is obtained.

As a preliminary, we define some constants that we will need, and set the Mathcad worksheet ORIGIN to 1 so that subscripts start at unity rather than at zero.

$DegPerRad := \frac{180}{\pi}$ $ORIGIN \equiv 1$

$SecPerDeg := 3600.0$ $SecPerRad := 206264.806$

$SecPerRev := SecPerDeg \cdot 360.0$

1. Retrieve the test case values from disk, as specified by worksheet HD1, or as specified by your own worksheet that was derived from HD1 by duplication and modification.

$n := \mathrm{READPRN}(\text{"NOBS.dat"})_1$ Number of observations.

$\mathbf{JDT} := \mathrm{READPRN}(\text{"TVALS.prn"})$ Observation times.

$\mathrm{Epoch} := \mathrm{READPRN}(\text{"EPOCH.dat"})_1$ Epoch of state vector solution.

$W := \mathrm{READPRN}(\text{"WEIGHTS.prn"})$ Measurement weights matrix.

$\mathbf{R} := \mathrm{READPRN}(\text{"RVALS.prn"})$ Values of **R**.

$Y := \mathrm{READPRN}(\text{"YVALS.prn"})$ Values of Y.

$X := \mathrm{READPRN}(\text{"STATE.prn"})$ State vector (corrected by HDC).

$\mathbf{RMS} := \mathrm{READPRN}(\text{"RMS.prn"})$ RMS history for state corrections by HDC (one entry for each iteration).

$\Delta := \mathrm{READPRN}(\text{"DELTA.prn"})$ Δ values for light-time correction.

$N := 2 \cdot n$ Set number of measurements.

$k := 0.01720209895$ Set Gaussian constant for heliocentric motion.

$\mu := 1$ Assume that mass of secondary (comet or asteroid) is negligible relative to mass of primary (Sun).

$K := k \cdot \sqrt{\mu}$

2. Define the procedural functions needed in the DC: **C**, **FG**, **GMAT**, and **FXA**.

For path propagation one needs to calculate only c_0 through c_3, but for the state transition matrix, G, one needs c_0 through c_5. To keep down the length of this worksheet we define one version of **C**, the one that calculates c_0 through c_5. (Remember that since the ORIGIN = 1, the subscripts of the c-functions that we will use outside of the function **C** will range from 1 through 6, rather than from 0 through 5.)

The material added below works in parallel with the main worksheet flow on the left to calculate and display a comparison plot of the ephemeris of Ceres at 30-day intervals starting at the $\mathbf{JDT}_1$ epoch.

What is compared is the ephemeris of Ceres obtained from the DC solution with 17 actual Piazzi observations vs. the DC solution with 19 Der ORBIT2-computed observations. There are 13 ephemeris comparison points that span 360 days past epoch.

Read values and points needed for comparison plot, HDC with 17 actual Piazzi obs vs. HDC with 19 Der ORBIT2-computed obs:

$\mathbf{R1} := \mathrm{READPRN}(\text{"RVALS1.prn"})$

$\mathbf{M2} := \mathrm{READPRN}(\text{"POINTS\_DER.prn"})$

Note on READPRN for **M2**:

POINTS_DER.prn has RA/DEC predictions from Hd1/Hdc solution with 17 Piazzi observations. The predictions are at epoch plus 12 additional points spaced at equal 30-day intervals past epoch.

POINTS_DER_ORBIT2.prn has RA/DEC predictions using ORBIT2 solar system numerical integration. Here, too, the predictions are at epoch plus 12 additional points spaced at equal 30-day intervals past epoch.

Input Gauss's search ephemeris as provided to von Zach. See *Monatliche Correspondenz,* Vol. 4, p. 647 (1801 December).

$\mathbf{Gauss} := \mathrm{READPRN}(\text{"GAUSS.prn"})$

$$\mathbf{Gauss} = \begin{pmatrix} 11.65585 & 12.50322 \\ 11.7885 & 12.06647 \\ 11.91407 & 11.68969 \\ 12.03162 & 11.38053 \\ 12.14167 & 11.155 \\ 12.24181 & 11.01156 \\ 12.3331 & 10.94381 \end{pmatrix}$$

$$
\mathbf{C}(x) := \left|
\begin{array}{l}
N \leftarrow 0 \\
\text{while } |x| \geq 0.1 \\
\quad \left| \begin{array}{l} x \leftarrow \dfrac{x}{4} \\ N \leftarrow N + 1 \end{array} \right. \\
c_5 \leftarrow \dfrac{\left[1 - \dfrac{x}{42}\left[1 - \dfrac{x}{72}\left[1 - \dfrac{x}{110}\left[1 - \dfrac{x}{156}\left[1 - \dfrac{x}{210}\left(1 - \dfrac{x}{272}\right)\right]\right]\right]\right]\right]}{120} \\
c_4 \leftarrow \dfrac{\left[1 - \dfrac{x}{30}\left[1 - \dfrac{x}{56}\left[1 - \dfrac{x}{090}\left[1 - \dfrac{x}{132}\left[1 - \dfrac{x}{182}\left(1 - \dfrac{x}{240}\right)\right]\right]\right]\right]\right]}{24} \\
c_3 \leftarrow \dfrac{1}{6} - c_5 \cdot x \\
c_2 \leftarrow \dfrac{1}{2} - c_4 \cdot x \\
c_1 \leftarrow 1 - c_3 \cdot x \\
c_0 \leftarrow 1 - c_2 \cdot x \\
\text{while } N > 0 \\
\quad \left| \begin{array}{l}
N \leftarrow N - 1 \\
c_5 \leftarrow \dfrac{\left(c_2 \cdot c_3 + c_4 + c_5\right)}{16} \\
c_4 \leftarrow \dfrac{\left(c_2 \cdot c_2 + c_4 + c_4\right)}{8} \\
c_3 \leftarrow \dfrac{\left(c_1 \cdot c_2 + c_3\right)}{4} \\
c_2 \leftarrow \dfrac{c_1 \cdot c_1}{2} \\
c_1 \leftarrow c_1 \cdot c_0 \\
c_0 \leftarrow 2 \cdot c_0 \cdot c_0 - 1
\end{array} \right. \\
\left(c_0 \;\; c_1 \;\; c_2 \;\; c_3 \;\; c_4 \;\; c_5\right)^T
\end{array}
\right.
$$

Function UKEP solves the uniform Kepler equation for function **FG**. **FG,** in turn, propagates position and velocity for function **FXA**.

$$
\mathrm{UKEP}(\tau, r_o, \sigma_o, \alpha) := \left|
\begin{array}{l}
s \leftarrow \dfrac{\tau}{r_o} \\
\Delta s \leftarrow s \\
\text{while } |\Delta s| \ge 0.00000001 \\
\quad \left|
\begin{array}{l}
\mathbf{c} \leftarrow \mathbf{C}(\alpha \cdot s^2) \\
F \leftarrow r_o \cdot s \cdot \mathbf{c}_2 + \sigma_o \cdot s^2 \cdot \mathbf{c}_3 + s^3 \cdot \mathbf{c}_4 - \tau \\
DF \leftarrow r_o \cdot \mathbf{c}_1 + \sigma_o \cdot s \cdot \mathbf{c}_2 + s^2 \cdot \mathbf{c}_3 \\
DDF \leftarrow \sigma_o \cdot \mathbf{c}_1 + (1 - r_o \cdot \alpha) \cdot s \cdot \mathbf{c}_2 \\
m \leftarrow 1 \quad \text{if } DF \ge 0 \\
m \leftarrow -1 \quad \text{otherwise} \\
\Delta s \leftarrow \dfrac{-5 \cdot F}{\left[DF + m \cdot \sqrt{\left|(4 \cdot DF)^2 - 20 \cdot F \cdot DDF\right|}\right]} \\
s \leftarrow s + \Delta s
\end{array}
\right. \\
s
\end{array}
\right.
$$

$$
\mathbf{FG}(K, \mathbf{r_o}, \mathbf{v_o}, \Delta t) := \left|
\begin{array}{l}
\tau \leftarrow K \cdot \Delta t \\
r_o \leftarrow \sqrt{\mathbf{r_o} \cdot \mathbf{r_o}} \\
\sigma_o \leftarrow \mathbf{r_o} \cdot \mathbf{v_o} \\
\alpha \leftarrow \dfrac{2}{r_o} - \mathbf{v_o} \cdot \mathbf{v_o} \\
s \leftarrow \mathrm{UKEP}(\tau, r_o, \sigma_o, \alpha) \\
\mathbf{c} \leftarrow \mathbf{C}(\alpha \cdot s^2) \\
f_r \leftarrow 1 - s^2 \cdot \mathbf{c}_3 \cdot r_o^{-1} \\
g_r \leftarrow \tau - s^3 \cdot \mathbf{c}_4 \\
r \leftarrow r_o \cdot \mathbf{c}_1 + \sigma_o \cdot s \cdot \mathbf{c}_2 + s^2 \cdot \mathbf{c}_3 \\
f_v \leftarrow -s \cdot \mathbf{c}_2 \cdot (r \cdot r_o)^{-1} \\
g_v \leftarrow 1 - s^2 \cdot \mathbf{c}_3 \cdot r^{-1} \\
\begin{pmatrix} K & \alpha & r_o & f_r & f_v \\ \tau & s & r & g_r & g_v \end{pmatrix}
\end{array}
\right.
$$

Function **GMAT** provides the state transition matrix for function **FXA**.

The state transition matrix formulation that we use below is based upon the seminal works of Goodyear [4], [5]. See also Shepperd [6], Battin [7], and Der [8] for more recent expositions.

Before defining **GMAT**, we define functions $\mathbf{S}_{11}$, $\mathbf{S}_{12}$, $\mathbf{S}_{21}$, and $\mathbf{S}_{22}$ just to make **GMAT** fit horizontally and vertically within the margins of a single Mathcad page.

$$\mathbf{S_{11}}(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}) := \begin{bmatrix} -\dfrac{f_v \cdot \mathbf{s}_2 + \dfrac{f_r - 1}{r_o}}{r_o} & -f_v \cdot \mathbf{s}_3 \\ \dfrac{(f_r - 1) \cdot \mathbf{s}_2}{r_o} & (f_r - 1) \cdot \mathbf{s}_3 \end{bmatrix}$$

$$\mathbf{S_{12}}(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}) := \begin{bmatrix} -f_v \cdot \mathbf{s}_3 & -(g_v - 1) \cdot \mathbf{s}_3 \\ (f_r - 1) \cdot \mathbf{s}_3 & g_r \cdot \mathbf{s}_3 \end{bmatrix}$$

$$\mathbf{S_{21}}(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}) := \begin{bmatrix} -f_v \cdot \left(\dfrac{\mathbf{s}_1}{r_o \cdot r} + \dfrac{1}{r^2} + \dfrac{1}{r_o^2} \right) & -\dfrac{f_v \cdot \mathbf{s}_2 + \dfrac{g_v - 1}{r}}{r} \\ \dfrac{f_v \cdot \mathbf{s}_2 + \dfrac{(f_r - 1)}{r_o}}{r_o} & f_v \cdot \mathbf{s}_3 \end{bmatrix}$$

$$\mathbf{S_{22}}(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}) := \begin{bmatrix} -\dfrac{f_v \cdot \mathbf{s}_2 + \dfrac{g_v - 1}{r}}{r} & \dfrac{-(g_v - 1) \cdot \mathbf{s}_2}{r} \\ f_v \cdot \mathbf{s}_3 & (g_v - 1) \cdot \mathbf{s}_3 \end{bmatrix}$$

(Note that because ORIGIN = 1, the subscripts of the c-functions and Goodyear's s-functions range from 1 to 6 rather than from 0 to 5. It is especially important to note this difference when checking the **GMAT** formulas against Goodyear's original works.)

$$
\mathbf{GMAT}(\mathbf{M},\mathbf{r_o},\mathbf{v_o},\mathbf{r},\mathbf{v}) := \left|
\begin{array}{l}
\tau \leftarrow \mathbf{M}_{2,1} \\
\alpha \leftarrow \mathbf{M}_{1,2} \\
s \leftarrow \mathbf{M}_{2,2} \\
r_o \leftarrow \mathbf{M}_{1,3} \\
r \leftarrow \mathbf{M}_{2,3} \\
f_r \leftarrow \mathbf{M}_{1,4} \\
g_r \leftarrow \mathbf{M}_{2,4} \\
f_v \leftarrow \mathbf{M}_{1,5} \\
g_v \leftarrow \mathbf{M}_{2,5} \\
\mathbf{c} \leftarrow \mathbf{C}\left(\alpha\cdot s^2\right) \\
\mathbf{s} \leftarrow \left(\mathbf{c}_1 \;\; s\cdot\mathbf{c}_2 \;\; s^2\cdot\mathbf{c}_3 \;\; s^3\cdot\mathbf{c}_4 \;\; s^4\cdot\mathbf{c}_5 \;\; s^5\cdot\mathbf{c}_6\right)^T \\
U \leftarrow \mathbf{s}_3\cdot\tau + s\cdot\mathbf{s}_5 - 3\cdot\mathbf{s}_6 \\
\mathbf{A} \leftarrow \text{augment}(\mathbf{r},\mathbf{v}) \\
\mathbf{B} \leftarrow \text{augment}\left(\mathbf{r_o},\mathbf{v_o}\right)^T \\
\mathbf{a_o} \leftarrow \dfrac{-\mathbf{r_o}}{r_o^{\,3}} \\
\mathbf{a} \leftarrow \dfrac{-\mathbf{r}}{r^3} \\
\mathbf{I} \leftarrow \text{identity}(3) \\
\mathbf{G_{11}} \leftarrow f_r\cdot\mathbf{I} + U\cdot\mathbf{v}\cdot\mathbf{a_o}^T + \mathbf{A}\cdot\mathbf{S_{11}}\left(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}\right)\cdot\mathbf{B} \\
\mathbf{G_{12}} \leftarrow g_r\cdot\mathbf{I} - U\cdot\mathbf{v}\cdot\mathbf{v_o}^T + \mathbf{A}\cdot\mathbf{S_{12}}\left(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}\right)\cdot\mathbf{B} \\
\mathbf{G_{21}} \leftarrow f_v\cdot\mathbf{I} + U\cdot\mathbf{a}\cdot\mathbf{a_o}^T + \mathbf{A}\cdot\mathbf{S_{21}}\left(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}\right)\cdot\mathbf{B} \\
\mathbf{G_{22}} \leftarrow g_v\cdot\mathbf{I} - U\cdot\mathbf{a}\cdot\mathbf{v_o}^T + \mathbf{A}\cdot\mathbf{S_{22}}\left(r_o, r, f_r, g_r, f_v, g_v, \mathbf{s}\right)\cdot\mathbf{B} \\
\text{stack}\left(\text{augment}\left(\mathbf{G_{11}},\mathbf{G_{12}}\right), \text{augment}\left(\mathbf{G_{21}},\mathbf{G_{22}}\right)\right)
\end{array}\right.
$$

Below is an example of a "trick" in Mathcad. Since calculation flow is from left to right, top to bottom, by making a copy of **FXA** and pasting it to the right of the original, I can modify the copy and Mathcad will use the modified copy instead of the original.

Function **FXA** calculates **FX**, the N-by-1computed measurements vector, and A, the N-by-6 A-matrix of partials of the measurements at time t_i with respect to the state at time t_o.

Temporarily disable light-time displacement in **FXA** in order to compare solution with that of HGM using ORBIT2-derived TOD observations.

$$
\mathbf{FXA}(K,\mathbf{r_o},\mathbf{v_o}) := \left|
\begin{array}{l}
\text{for } i \in 1..n \\
\quad \left|
\begin{array}{l}
\mathbf{M} \leftarrow \mathbf{FG}\left(K,\mathbf{r_o},\mathbf{v_o},\mathbf{JDT}_i - \text{Epoch} - 0.005768\cdot\Delta_{2\cdot i-1}\right) \\
\tau \leftarrow \mathbf{M}_{2,1} \\
\alpha \leftarrow \mathbf{M}_{1,2} \\
s \leftarrow \mathbf{M}_{2,2} \\
r_o \leftarrow \mathbf{M}_{1,3} \\
r \leftarrow \mathbf{M}_{2,3} \\
f_r \leftarrow \mathbf{M}_{1,4} \\
g_r \leftarrow \mathbf{M}_{2,4} \\
f_v \leftarrow \mathbf{M}_{1,5} \\
g_v \leftarrow \mathbf{M}_{2,5} \\
\mathbf{r} \leftarrow f_r\cdot\mathbf{r_o} + g_r\cdot\mathbf{v_o} \\
\mathbf{v} \leftarrow f_v\cdot\mathbf{r_o} + g_v\cdot\mathbf{v_o} \\
\boldsymbol{\rho} \leftarrow \mathbf{r} + \mathbf{R}^{\langle i \rangle} \\
\rho \leftarrow \sqrt{\boldsymbol{\rho}\cdot\boldsymbol{\rho}} \\
j \leftarrow 2\cdot i - 1 \\
k \leftarrow j + 1 \\
RA \leftarrow \text{angle}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2\right) \\
DEC \leftarrow \text{asin}\left(\boldsymbol{\rho}_3\cdot\rho^{-1}\right) \\
\mathbf{FX}_j \leftarrow \cos\left(Y_k\right)\cdot RA \\
\mathbf{FX}_k \leftarrow DEC \\
\boldsymbol{\Delta}_j \leftarrow \rho \\
\boldsymbol{\Delta}_k \leftarrow 0 \\
O \leftarrow \frac{1}{\rho}\cdot\begin{pmatrix} -\sin(RA) & \cos(RA) & 0 & 0 & 0 & 0 \\ -\sin(DEC)\cdot\cos(RA) & -\sin(DEC)\cdot\sin(RA) & \cos(DEC) & 0 & 0 & 0 \end{pmatrix} \\
G \leftarrow \mathbf{GMAT}\left(\mathbf{M},\mathbf{r_o},\mathbf{v_o},\mathbf{r},\mathbf{v}\right) \\
A \leftarrow O\cdot G \ \text{ if } i = 1 \\
A \leftarrow \text{stack}(A, O\cdot G) \ \text{ otherwise}
\end{array}\right. \\
A \leftarrow \text{augment}(\mathbf{FX}, A) \\
\text{augment}(\boldsymbol{\Delta}, A)
\end{array}\right.
$$

$$
\mathbf{FXA}(K,\mathbf{r_o},\mathbf{v_o}) := \left|
\begin{array}{l}
\text{for } i \in 1..n \\
\quad \left|
\begin{array}{l}
\mathbf{M} \leftarrow \mathbf{FG}\left(K,\mathbf{r_o},\mathbf{v_o},\mathbf{JDT}_i - \text{Epoch}\right) \\
\tau \leftarrow \mathbf{M}_{2,1} \\
\alpha \leftarrow \mathbf{M}_{1,2} \\
s \leftarrow \mathbf{M}_{2,2} \\
r_o \leftarrow \mathbf{M}_{1,3} \\
r \leftarrow \mathbf{M}_{2,3} \\
f_r \leftarrow \mathbf{M}_{1,4} \\
g_r \leftarrow \mathbf{M}_{2,4} \\
f_v \leftarrow \mathbf{M}_{1,5} \\
g_v \leftarrow \mathbf{M}_{2,5} \\
\mathbf{r} \leftarrow f_r\cdot\mathbf{r_o} + g_r\cdot\mathbf{v_o} \\
\mathbf{v} \leftarrow f_v\cdot\mathbf{r_o} + g_v\cdot\mathbf{v_o} \\
\boldsymbol{\rho} \leftarrow \mathbf{r} + \mathbf{R}^{\langle i \rangle} \\
\rho \leftarrow \sqrt{\boldsymbol{\rho}\cdot\boldsymbol{\rho}} \\
j \leftarrow 2\cdot i - 1 \\
k \leftarrow j + 1 \\
RA \leftarrow \text{angle}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2\right) \\
DEC \leftarrow \text{asin}\left(\boldsymbol{\rho}_3\cdot\rho^{-1}\right) \\
\mathbf{FX}_j \leftarrow \cos\left(Y_k\right)\cdot RA \\
\mathbf{FX}_k \leftarrow DEC \\
\boldsymbol{\Delta}_j \leftarrow \rho \\
\boldsymbol{\Delta}_k \leftarrow 0 \\
O \leftarrow \frac{1}{\rho}\cdot\begin{pmatrix} -\sin(RA) & \cos(RA) & 0 & 0 & 0 & 0 \\ -\sin(DEC)\cdot\cos(RA) & -\sin(DEC)\cdot\sin(RA) & \cos(DEC) & 0 & 0 & 0 \end{pmatrix} \\
G \leftarrow \mathbf{GMAT}\left(\mathbf{M},\mathbf{r_o},\mathbf{v_o},\mathbf{r},\mathbf{v}\right) \\
A \leftarrow O\cdot G \ \text{ if } i = 1 \\
A \leftarrow \text{stack}(A, O\cdot G) \ \text{ otherwise}
\end{array}\right. \\
A \leftarrow \text{augment}(\mathbf{FX}, A) \\
\text{augment}(\boldsymbol{\Delta}, A)
\end{array}\right.
$$

3. Obtain the computed measurements, **FX**, and the A-matrix, A, by invoking **FXA**.

$$\mathbf{r_o} := \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \qquad \mathbf{v_o} := \begin{pmatrix} X_4 \\ X_5 \\ X_6 \end{pmatrix} \cdot \frac{1}{K}$$

$$\mathbf{M} := \mathbf{FXA}(K, \mathbf{r_o}, \mathbf{v_o})$$

$$\mathbf{FX} := \mathbf{M}^{\langle 2 \rangle} \qquad A := \text{submatrix}(\mathbf{M}, 1, N, 3, 8)$$

Extract the geocentric distance values as needed for the light-time correction.

$$\boldsymbol{\Delta} := \mathbf{M}^{\langle 1 \rangle}$$

(To see all entries of **FX** or A, click on the column vector or matrix, respectively, and scroll up/down or right/left.)

FX =

	1
1	0.87061649
2	0.27277156
3	0.86913415
4	0.27374488
5	0.86651011
6	0.27575906
7	0.86130628
8	0.28232962
9	0.86005835
10	0.28713294
11	0.86095872
12	0.29358074
13	0.86206144
14	0.29628839
15	0.86277151
16	...

A =

	1	2
1	-0.40491515	0.31866265
2	-0.08584821	-0.10908477
3	-0.40257292	0.31766557
4	-0.08587629	-0.10883056
5	-0.3979437	0.31547555
6	-0.0858876	-0.1083467
7	-0.38459127	...

Define function to calculate geocentric positions of Ceres at 30-day intervals starting at epoch. Need to input **R1** array of heliocentric Earth positions, computed in HD1, in order for this to work. Also need **M2** array from the HD1/HDC worksheet pair, with 19 Der ORBIT2-computed positions as needed for the comparison plot.

```
Points(K, r_o, v_o) :=  for j ∈ 1..13
                           t_j ← JDT_1 − Epoch + 30·(j − 1)
                           M ← FG(K, r_o, v_o, t_j)
                           τ ← M_{2,1}
                           α ← M_{1,2}
                           s ← M_{2,2}
                           r_o ← M_{1,3}
                           r ← M_{2,3}
                           f_r ← M_{1,4}
                           g_r ← M_{2,4}
                           f_v ← M_{1,5}
                           g_v ← M_{2,5}
                           r ← f_r·r_o + g_r·v_o
                           v ← f_v·r_o + g_v·v_o
                           ρ ← r + R1^⟨j⟩
                           ρ ← √(ρ·ρ)
                           RA ← angle(ρ_1, ρ_2)·DegPerRad
                           DEC ← asin(ρ_3·ρ^(−1))·DegPerRad
                           A ← (t_j   t_j + JDT_1   RA/15   DEC)  if j = 1
                           A ← stack[A, (t_j   t_j + JDT_1   RA/15   DEC)]  otherwise
                        A
```

$$\mathbf{M1} := \mathbf{Points}(K, \mathbf{r_o}, \mathbf{v_o})$$

4. Compute the residuals, ΔY, the $A^T WA$ matrix ATWA, and the $A^T W \Delta Y$ matrix, ATWΔY.

$$\Delta Y := Y - \mathbf{FX}$$

$$\mathrm{ATWA} := A^T \cdot W \cdot A$$

$$\mathrm{ATW}\Delta Y := A^T \cdot W \cdot \Delta Y$$

$\Delta Y =$

	1
1	-0.00001434
2	0.00000159
3	0.00001083
4	0.00000759
5	-0.0000027
6	-0.00000868
7	0.00000676
8	-0.00001292
9	0.00001459
10	0.00000136
11	-0.00000565
12	0.00000352
13	-0.00000121
14	0.00000986
15	-0.0000196
16	...

(To see all entries of ΔY, click on the column vector and scroll up/down or right/left.)

$$\mathrm{ATWA} = \begin{pmatrix} 2.351508 & -1.5972887 & -0.6199579 & 0.8197251 & -0.5420871 & -0.2174363 \\ -1.5972887 & 1.5580222 & -0.7910734 & -0.5460656 & 0.5332222 & -0.2825486 \\ -0.6199579 & -0.7910734 & 3.2855841 & -0.2189415 & -0.2823915 & 1.1231481 \\ 0.8197251 & -0.5460656 & -0.2189415 & 0.4007607 & -0.2619893 & -0.1061408 \\ -0.5420871 & 0.5332222 & -0.2823915 & -0.2619893 & 0.2561328 & -0.1392604 \\ -0.2174363 & -0.2825486 & 1.1231481 & -0.1061408 & -0.1392604 & 0.5430261 \end{pmatrix}$$

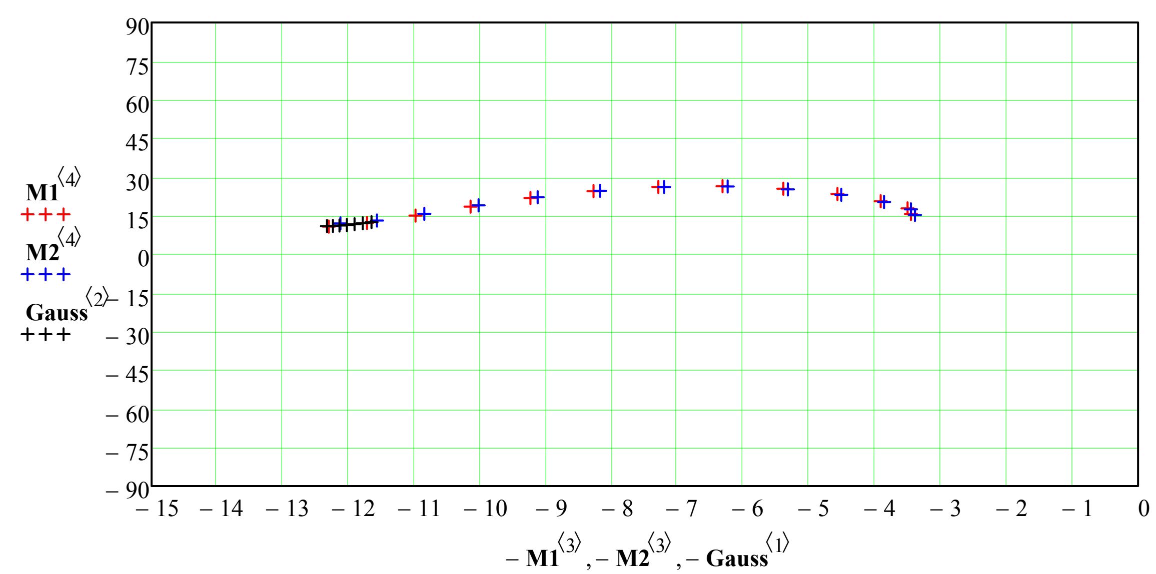
$$\mathrm{ATW}\Delta Y = \begin{pmatrix} 0 \\ 1.0733 \times 10^{-15} \\ -0 \\ -0 \\ -0 \\ 0 \end{pmatrix}$$

M1 points represent HDC solution for 17 actual Piazzi obs. **M2** points represent HDC solution for 19 Der ORBIT2-computed Piazzi obs:

$$\mathbf{M1} = \begin{pmatrix} 0 & 2378862.3634 & 3.453183 & 15.628659 \\ 30 & 2378892.3634 & 3.505395 & 17.813179 \\ 60 & 2378922.3634 & 3.914105 & 20.634205 \\ 90 & 2378952.3634 & 4.571208 & 23.38934 \\ 120 & 2378982.3634 & 5.392999 & 25.475214 \\ 150 & 2379012.3634 & 6.318119 & 26.466655 \\ 180 & 2379042.3634 & 7.295335 & 26.147587 \\ 210 & 2379072.3634 & 8.280609 & 24.529099 \\ 240 & 2379102.3634 & 9.239695 & 21.839429 \\ 270 & 2379132.3634 & 10.148411 & 18.489777 \\ 300 & 2379162.3634 & 10.986372 & 15.041401 \\ 330 & 2379192.3634 & 11.724098 & 12.190926 \\ 360 & 2379222.3634 & 12.304506 & 10.752828 \end{pmatrix}$$

$$\mathbf{M2} = \begin{pmatrix} 0 & 2378862.3634 & 3.39371 & 15.24174 & 3.39371 & 15.2413 \\ 30 & 2378892.3634 & 3.45324 & 17.43789 & 3.45324 & 17.43758 \\ 60 & 2378922.3634 & 3.86198 & 20.27967 & 3.86199 & 20.27941 \\ 90 & 2378952.3634 & 4.5134 & 23.06853 & 4.51346 & 23.06808 \\ 120 & 2378982.3634 & 5.32587 & 25.21261 & 5.32605 & 25.21175 \\ 150 & 2379012.3634 & 6.23988 & 26.29566 & 6.24026 & 26.29404 \\ 180 & 2379042.3634 & 7.20579 & 26.10347 & 7.20645 & 26.10046 \\ 210 & 2379072.3634 & 8.18052 & 24.64158 & 8.18154 & 24.63619 \\ 240 & 2379102.3634 & 9.12977 & 22.12891 & 9.13118 & 22.11999 \\ 270 & 2379132.3634 & 10.02824 & 18.96953 & 10.03011 & 18.95587 \\ 300 & 2379162.3634 & 10.85363 & 15.72572 & 10.85606 & 15.70595 \\ 330 & 2379192.3634 & 11.57367 & 13.10856 & 11.57685 & 13.08091 \\ 360 & 2379222.3634 & 12.1273 & 11.96112 & 12.13152 & 11.92318 \end{pmatrix}$$

Sky plot of right ascension, in hours, vs. declination, in degrees, for HD1/HDC solution with 17 Piazzi observations (red plusses) vs. HD1/HDC solution with 19 Der ORBIT2-computed Piazzi observations (blue plusses):

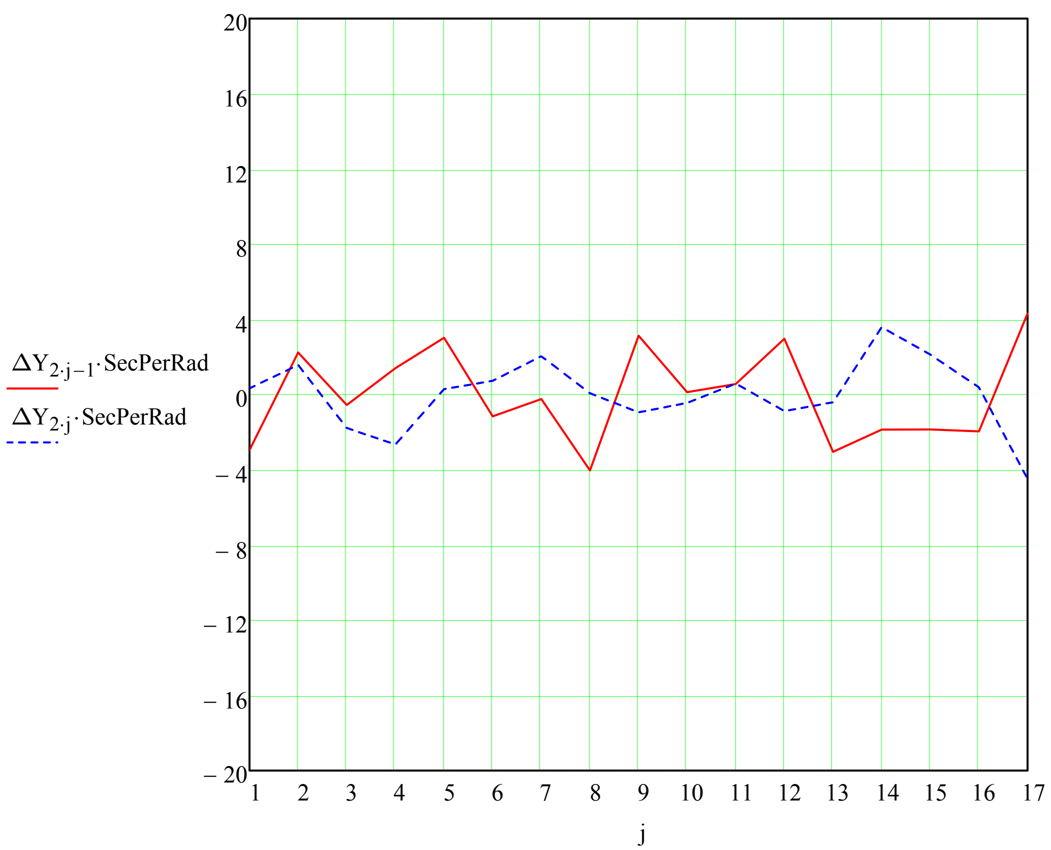


(See also the plot at the end of this worksheet.)

Note: Since right ascension increases from 0 to 24 hours west to east in the sky, and right to left on a sky chart, **M1** and **M2** were made negative and the horizontal axes go from 0 to -15 hours so that the path of Ceres is correct when superimposed on a star chart.

Plot the Δα cos δ (red trace in the plot) and Δδ residuals (blue trace in the plot) for n = 17 observations. The units are arc-seconds for both traces.

$j := 1 .. n$



Note: this plot, and indeed all of the calculations in the worksheet, are redone for every iteration of the DC, i.e., at every Ctrl-F9 or Tools Menu > Calculate > Calculate Worksheet click.

Right ascension differences, in degrees:

$$\left(\mathbf{M1}^{\langle 3\rangle} - \mathbf{M2}^{\langle 3\rangle}\right)\cdot 15 = \begin{pmatrix} 0.89210113 \\ 0.78232816 \\ 0.78188813 \\ 0.86713482 \\ 1.00692924 \\ 1.17363085 \\ 1.34322703 \\ 1.50127529 \\ 1.64885196 \\ 1.80252724 \\ 1.99116541 \\ 2.25635735 \\ 2.65806368 \end{pmatrix}$$

Declination differences, in degrees:

$$\mathbf{M1}^{\langle 3\rangle} - \mathbf{M2}^{\langle 3\rangle} = \begin{pmatrix} 0.05947341 \\ 0.05215521 \\ 0.05212588 \\ 0.05780899 \\ 0.06712862 \\ 0.07824206 \\ 0.08954847 \\ 0.10008502 \\ 0.10992346 \\ 0.12016848 \\ 0.13274436 \\ 0.15042382 \\ 0.17720425 \end{pmatrix}$$

Conclusion from these right-side-of-the-worksheet side calculations:

The orbital solution in the worksheet

Hdc_Ceres_1801_Piazzi_17_obs_Theta=RA_plot.xmcd

differs from the orbital solution in the worksheet

Hdc_Ceres_1801_Der_19_obs_Theta=RA.xmcd

by about 2.66 degrees of right ascension after 360 days.

It is important to note that, as can be seen from the comparison plot above, both solutions follow the same "corridor" in the sky. They differ mainly in predicting precisely where Ceres was along its path in this corridor.

5. Solve for and apply the corrections to state, ΔX. Compute the current RMS error, display the RMS error history, and test for convergence.

$$\Delta X := ATWA^{-1} \cdot ATW\Delta Y \qquad \Delta X = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0 \\ -0 \\ -0 \end{pmatrix}$$

$$X := \mathrm{stack}(\mathbf{r_o}, \mathbf{v_o}) + \Delta X$$

$$WSS := \sum_{i=1}^{N} \left(W_{i,i} \cdot \Delta Y_i\right)^2$$

Weighted sum of squares of residuals.

$$WSS = 0$$

$$WRMS := SecPerRad \cdot \sqrt{\frac{WSS}{N}}$$

Weighted RMS in arcsec.

$$WRMS = 2.15526$$

$$PWSS := \sum_{i=1}^{6} \left(ATW\Delta Y_i \cdot \Delta X_i\right)$$

Predicted weighted sum of squares of residuals for next iteration.

$$PWSS = 0$$

$$PWRMS := SecPerRad \cdot \sqrt{\frac{|WSS - PWSS|}{N}}$$

Predicted weighted RMS for next iteration, in arc-sec.

$$PWRMS = 2.15526$$

$$Converged := \begin{cases} 1 & \text{if } |WRMS - PWRMS| < 0.01 \cdot WRMS \\ 0 & \text{otherwise} \end{cases}$$

$$Converged = 1$$

In the Addendum to my AMOS 2016 paper with Gim J. Der, "Reconstruction of the 1801 Discovery Orbit of Ceres via Contemporary Angles-Only Algorithms," we show that Gauss's IOD solution with the three Piazzi observations dated 1801 January 1, January 21, and February 11 is somewhat better than the HDC solution in this HDC worksheet that uses the 17 best Piazzi observations.

But it is further shown in the Addendum that when HDC is run with the exact same three, and only three observations that Gauss used, then our HDC solution now in 2016 is slightly better than that of Gauss in 1801.

"Slightly better" is attributed to the fact that we have a better solar ephemeris model in 2016 than Gauss had available to him in 1801.

$\text{APPENDPRN}(\text{"RMS.prn"}) := (\text{WRMS} \quad \text{Converged})$

$\mathbf{RMS} := \text{READPRN}(\text{"RMS.prn"})$

RMS History: $\mathbf{RMS} = \begin{pmatrix} 0 & 0 \\ 3.13 & 0 \\ 2.155 & 1 \\ 2.155 & 1 \end{pmatrix}$

Number of iterations:

$\text{Iterations} := \text{rows}(\mathbf{RMS}) - 1$

$\text{Iterations} = 3$

6. Write the corrected state vector to disk and convert to conic elements.

$$\text{WRITEPRN}(\text{"STATE.prn"}) := \text{stack}\left[\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \begin{pmatrix} X_4 \\ X_5 \\ X_6 \end{pmatrix} \cdot K\right]$$

$\text{WRITEPRN}(\text{"DELTA.prn"}) := \boldsymbol{\Delta}$ (Also save geocentric distances for use in light-time corrections.)

First transform $\mathbf{r}_1$ and $\mathbf{v}_1$ from the HCI (HelioCentric Inertial) equatorial reference frame of date to the HCI ecliptic reference frame of date.

We will need the obliquity of the ecliptic, ε, at date of first observation, in order to transform the ECI ecliptic coordinates of date to ECI equatorial coordinates of date.

$$\varepsilon := \frac{23.4392911 - 0.0000004 \cdot (\mathbf{JDT}_1 - 2451545.0)}{\text{DegPerRad}} \qquad \mathbf{MO} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & -\sin(\varepsilon) \\ 0 & \sin(\varepsilon) & \cos(\varepsilon) \end{pmatrix}$$

$\mathbf{ECEQ}(\mathbf{r}) := \mathbf{MO} \cdot \mathbf{r}$ (Transforms from ecliptic to equatorial.)

$\mathbf{EQEC}(\mathbf{r}) := \mathbf{MO}^{-1} \cdot \mathbf{r}$ (Transforms from equatorial to ecliptic.)

$$\mathbf{r_1} := \mathbf{EQEC}\left(\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right) \qquad \mathbf{r_1} = \begin{pmatrix} 0.96711 \\ 2.52033 \\ -0.10532 \end{pmatrix}$$

$$\mathbf{v_1} := \mathbf{EQEC}\left[\begin{pmatrix} X_4 \\ X_5 \\ X_6 \end{pmatrix}\cdot K\right] \qquad \mathbf{v_1} = \begin{pmatrix} -0.00999 \\ 0.00297 \\ 0.00194 \end{pmatrix}$$

Need heliocentric state vectors for precession worksheet,

Precess_Ceres_1801_Elements_to_J2000.xmcd

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.96710782 \\ 2.35379252 \\ 0.90709088 \end{pmatrix} \qquad \begin{pmatrix} X_4 \\ X_5 \\ X_6 \end{pmatrix}\cdot K = \begin{pmatrix} -0.00998828 \\ 0.00194961 \\ 0.00295711 \end{pmatrix}$$

PVCO invokes function SCAL1, which we define now.

$$\mathrm{SCAL1}(K,\alpha,q,e,\upsilon) := \left| \begin{array}{l} \text{if } \alpha > 0 \\ \quad \left| \begin{array}{l} E \leftarrow \upsilon - 2\cdot\operatorname{atan}\left(\dfrac{e\cdot\sin(\upsilon)}{1+\sqrt{1-e^2}+e\cdot\cos(\upsilon)}\right) \\ s \leftarrow \dfrac{E}{\sqrt{\alpha}} \end{array}\right. \\ \text{otherwise} \\ \quad \left| \begin{array}{l} w \leftarrow \dfrac{1}{K}\cdot\sqrt{\dfrac{q}{1+e}}\cdot\tan\left(\dfrac{\upsilon}{2}\right) \\ s \leftarrow 2\cdot w \ \text{ if } \alpha = 0 \\ \text{otherwise} \\ \quad \left| \begin{array}{l} E \leftarrow 2\cdot\operatorname{atanh}\left(\sqrt{-\alpha}\cdot w\right) \\ s \leftarrow \dfrac{E}{\sqrt{-\alpha}} \end{array}\right. \end{array}\right. \\ s \end{array}\right.$$

Finally, now, we define function **PVCO**.

(Note that in **PVCO**, as defined in this document, the subscripts of the **P**, **Q**, and **W** vectors range from 1 through 3 rather than from 0 through 2. Also, the subscripts of **c** range from 1 through 4 rather than from 0 through 3.)

$$
\mathbf{PVCO}(K,\mathbf{r},\mathbf{v}) := \left|
\begin{array}{l}
r \leftarrow \sqrt{\mathbf{r}\cdot\mathbf{r}} \\
\mathbf{h} \leftarrow \mathbf{r} \times \mathbf{v} \\
h \leftarrow \sqrt{\mathbf{h}\cdot\mathbf{h}} \\
\mathbf{W} \leftarrow \dfrac{\mathbf{h}}{h} \\
\mathsf{E} \leftarrow \dfrac{\mathbf{v}\cdot\mathbf{v}}{2} - \dfrac{K^2}{r} \\
\alpha \leftarrow -2\cdot\mathsf{E} \\
p \leftarrow \dfrac{h^2}{K^2} \\
e \leftarrow \sqrt{1.0 - \alpha\cdot p\cdot K^{-2}} \\
q \leftarrow \dfrac{p}{1+e} \\
\mathbf{U} \leftarrow \dfrac{\mathbf{r}}{r} \\
\mathbf{V} \leftarrow \mathbf{W} \times \mathbf{U} \\
\upsilon \leftarrow \mathrm{angle}\left(\dfrac{h}{K^2}\cdot\mathbf{v}\cdot\mathbf{V} - 1.0, \dfrac{h}{K^2}\cdot\mathbf{v}\cdot\mathbf{U}\right) \\
\mathbf{P} \leftarrow \cos(\upsilon)\cdot\mathbf{U} - \sin(\upsilon)\cdot\mathbf{V} \\
\mathbf{Q} \leftarrow \sin(\upsilon)\cdot\mathbf{U} + \cos(\upsilon)\cdot\mathbf{V} \\
i \leftarrow \mathrm{acos}\left(\mathbf{W}_3\right) \\
\Omega \leftarrow \mathrm{angle}\left(-\mathbf{W}_2, \mathbf{W}_1\right) \\
\omega \leftarrow \mathrm{angle}\left(\mathbf{Q}_3, \mathbf{P}_3\right) \\
s \leftarrow \mathrm{SCAL1}(K,\alpha,q,e,\upsilon) \\
\mathbf{c} \leftarrow \mathbf{C}\left(\alpha\cdot s^2\right) \\
\Delta t \leftarrow q\cdot s + K^2\cdot e\cdot s^3\cdot\mathbf{c}_4 \\
\begin{pmatrix} q \\ e \\ i\cdot\mathrm{DegPerRad} \\ \Omega\cdot\mathrm{DegPerRad} \\ \omega\cdot\mathrm{DegPerRad} \\ \Delta t \end{pmatrix}
\end{array}
\right.
$$

We now invoke **PVCO** and place its output into array **CONIC**.

$$\mathbf{CONIC} := \mathbf{PVCO}\left(K, \mathbf{r_1}, \mathbf{v_1}\right)$$

$$\mathbf{CONIC} = \begin{pmatrix} 2.53024 \\ 0.08717 \\ 10.61659 \\ 81.02084 \\ 65.71636 \\ 1365.7595 \end{pmatrix}$$

We should note that the position vector input to **PVCO** must have units of A.U. and the velocity vector must have units of A.U. per day. We summarize our batch least squares DC's orbital solution as follows.

$\mathbf{CONIC}_1 = 2.53024365$	Perihelion distance in A.U.
$\mathbf{CONIC}_2 = 0.08716516$	Path eccentricity.
$\mathbf{CONIC}_3 = 10.61658703$	Path inclination, in degrees.
$\mathbf{CONIC}_4 = 81.0208356$	Celestial longitude of ascending node, in degrees.
$\mathbf{CONIC}_5 = 65.71636094$	Argument of perihelion, in degrees.
$\mathbf{CONIC}_6 = 1365.75950359$	Time of flight from perihelion to epoch, in days.

$a := \dfrac{\mathbf{CONIC}_1}{1 - \mathbf{CONIC}_2}$	$a = 2.77185262$	A.U.
$n_c := K \cdot a^{\frac{-3}{2}} \cdot \mathrm{DegPerRad}$	$n_c = 0.21357424$	deg/day
$M := \mathrm{mod}\left(n_c \cdot \mathbf{CONIC}_6, 360\right)$	$M = 291.6910488$	degrees
$P := \dfrac{2 \cdot \pi \cdot \mathrm{DegPerRad}}{n_c}$	$P = 1685.59653539$	days

$$\frac{P}{365.25} = 4.6149118$$

Period in Julian years.

Here now is a summary of our batch least squares, two-body DC solution, using 17 actual Piazzi observations over the time span 1801 January 1 - February 11, along with the HGM-Heliocentric ((HH1/HHC) solution using these same 17 observations.

Orbital Element/Parameter	HGM-Heliocentric Value	Batch DC Value
Semimajor axis, A.U.	2.75472901	2.77185262
Eccentricity	0.08078932	0.08716516
Inclination*, deg	10.6099206	10.61659
Longitude of Asc. Node*, deg	81.0379081	81.02084
Argument of perihelion*, deg	67.63590985	65.71636
Mean anomaly, deg*	290.7618546	291.6910488
Mean motion, deg/day	0.215568723	0.21357424
Orbital period, days	1670.00107878	1685.59653539
Orbital period, Julian years	4.57221377	4.6149118

*Angles are referred to true ecliptic and equinox of 1801 January 1.

Note 1: Contemporary period of Ceres in 2016, based upon mean daily motion of 0.2140 deg/day is:

$$P := \frac{2 \cdot \pi \cdot \mathrm{DegPerRad}}{0.2140} \qquad P = 1682.24299$$

$$\frac{P}{365.25} = 4.60573$$

Note 2: Batch DC solution includes light-time correction that is not present in the HGM-Heliocentric solution.

(Light-time displacement may have been disabled** in **FXA** in order to make direct comparison with ORBIT2-generated elements.)

Light-time displacement was indeed disabled in this worksheet, by redefining **FXA with the displacement term omitted. The modified **FXA** was placed to the right of the original **FXA**.

7. Repeat Steps 1-6, by clicking on "Calculate Worksheet", until convergence is obtained.

REFERENCES

[1] Mansfield, Roger L., "Algorithms for Reducing Radar Observations of a Hyperbolic Near Earth Flyby," *Journal of the Astronautical Sciences* (April-June 1993), pp. 249-259.

[2] Mansfield, Roger L., "Tracking Data Reduction for the Geotail, Mars Observer, and Galileo Missions," Eleventh Space Surveillance Workshop, MIT Lincoln Laboratory (March 31, 1993).

[Relevant material was distributed in a 19-page presentation handout, separately from the published conference proceedings, and was distributed only to attendees of the author's actual presentation. The published conference proceedings contain a six-page extended abstract of the presentation only (no equations).]

[3] Danby, J.M.A., *Fundamentals of Celestial Mechanics,* Willmann-Bell (2nd Ed. 1988), Appendix I.

[4] Goodyear, William H., "Completely General Closed-Form Solution for the Coordinates and Partial Derivatives of the Two-Body Problem," *Astronomical Journal,* Vol. 70, No. 3 (1965), pp. 189-192.

[5] Goodyear, William H., "A General Method for the Computation of Cartesian Coordinates and Partial Derivatives of the Two-Body Problem," NASA Report CR-522 (1966).

[This report, 97 pages in length, was prepared under Contract No. NAS 5-9762 by IBM for NASA's Goddard Space Flight Center, Greenbelt, Maryland.]

[6] Shepperd, Stanley W., "Universal Keplerian State Transition Matrix," *Celestial Mechanics,* Vol. 35, No. 2 (February 1985), pp. 129-144.

[7] Battin, Richard C., *An Introduction to the Mathematics and Methods of Astrodynamics,* AIAA Education Series (1987), Section 9.5.

[8] Der, Gim J., "An Elegant State Transition Matrix," *Journal of the Astronautical Sciences* (October - December 1997), pp. 371-390.

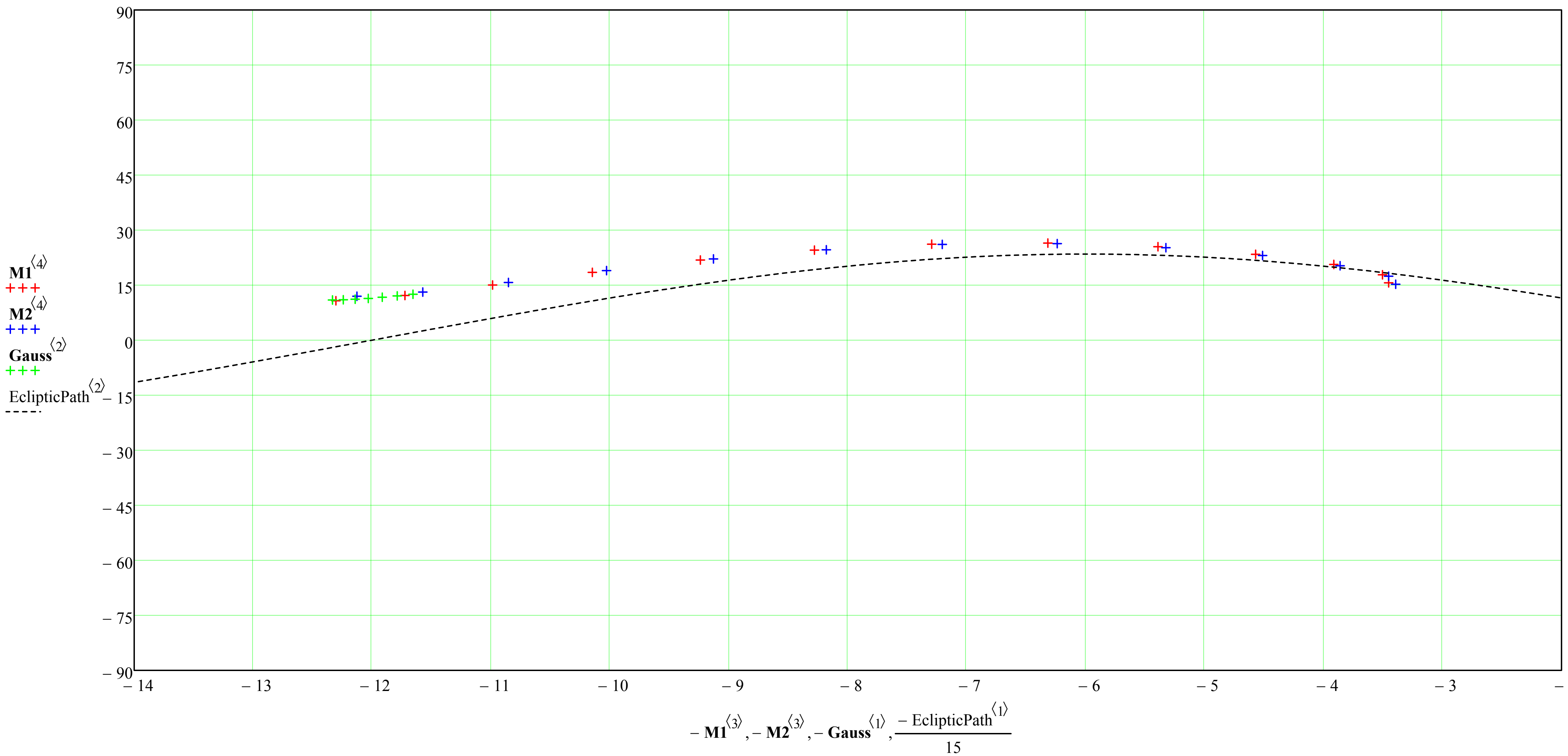
[9] Mansfield, Roger L., *Topics in Astrodynamics,* Astronomical Data Service, Colorado Springs, Colorado, February 28, 2003. Chapter 14 presents the Uniform Path Mechanics (UPM) equations of universal-variables, two-body orbital motion. Chapter 15 presents the author's treatment of batch least squares UPM differential correction ("Batch UPM DC") for Earth-orbital motion.

Define function needed to plot ecliptic path of Sun in 1801.

$$
\mathrm{Ecliptic2D} := \left|
\begin{array}{l}
\mathrm{Obl} \leftarrow \dfrac{23.46836}{\mathrm{DegPerRad}} \\
M \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\mathrm{Obl}) & -\sin(\mathrm{Obl}) \\ 0 & \sin(\mathrm{Obl}) & \cos(\mathrm{Obl}) \end{pmatrix} \\
\text{for } i \in 1..361 \\
\quad \left|
\begin{array}{l}
\alpha_i \leftarrow \dfrac{(i-1)}{\mathrm{DegPerRad}} \\
\delta_i \leftarrow 0 \\
V_1 \leftarrow \cos(\delta_i)\cdot\cos(\alpha_i) \\
V_2 \leftarrow \cos(\delta_i)\cdot\sin(\alpha_i) \\
V_3 \leftarrow \sin(\delta_i) \\
V \leftarrow M\cdot V \\
X_i \leftarrow V_1 \\
Y_i \leftarrow V_2 \\
Z_i \leftarrow V_3 \\
\alpha_i \leftarrow \alpha_i\cdot \mathrm{DegPerRad} \\
\delta_i \leftarrow \mathrm{asin}(Z_i)\cdot \mathrm{DegPerRad}
\end{array}
\right. \\
\mathrm{augment}(\alpha, \delta)
\end{array}
\right.
$$

$$\mathrm{EclipticPath} := \mathrm{Ecliptic2D}$$

Plot of 1801 Discovery Orbit of Ceres from 17-Best-Piazzi-Observations HDC Solution, ORBIT2 Numerically-Integrated Solution, and Gauss Predictions *from Monatliche Correspondenz,* Vol. 4 (December 1801, p. 647)



Plot is declination (vertical axis, degrees) vs. right ascension (horizontal axis, hours). Right ascensions are plotted "backward" above because right ascensions increase from right to left on a star chart. So just disregard the minus signs on the right ascension labels. Red plusses = Hd1_Hdc ephemeris with 17 Piazzi observations. Blue plusses = ORBIT2 ephemeris.Green plusses = Gauss ephemeris as published in *Monatliche Correspondenz,* Vol. 4, Article LVII, p. 647.